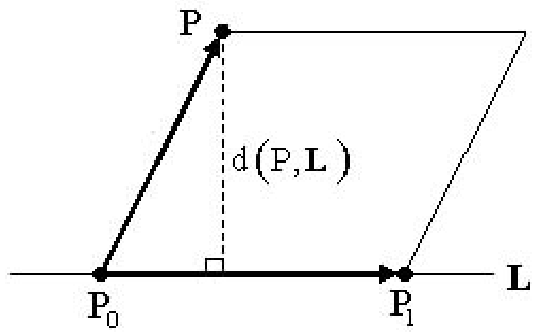
Math 2A – Vector Calculus – Final Exam – fall '09 Name________________________________
Show your work for credit. Do not use a calculator. Write all responses on separate paper.

1. Derive the formula for the distance d(P,L) between a point $P(x,y,0)$ and a line L through $P_0(x_0,y_0,0)$ and $P_1(x_1,y_1,0)$ in 3D by the following steps:
   a. Compute $\left|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P}\right|$ in terms of x, y, x_0, y_0, and x_1, y_1.
   b. Use the fact that the area of the parallelogram with edges $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P}$ is base*height to find a formula for the distance from P to L.



2. Consider the curve $\vec{r}(t) = \langle 5\sin t, 12t, 5\cos t \rangle$
   a. Find the unit tangent vector as a function of t.
   b. Find the curvature at $t = 0$.
   c. Find a vector parallel to the normal to the curve at $t = 0$.
   d. Parameterize the osculating circle at $t = 0$.

3. Consider the ellipsoid $4x^2 + 4y^2 + z^2 = 4$.
   a. Show that this surface is described by the spherical function $\rho = \dfrac{2}{\sqrt{1+3\sin^2\phi}}$

      Hint: $4x^2 + 4y^2 + z^2 = (x^2 + 3x^2) + (y^2 + 3y^2) + z^2 = 4$.
   b. Find a parameterization of the ellipsoid surface in terms of spherical coordinates θ and ϕ.
      Hint: recall that $\langle x, y, z \rangle = \langle \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi \rangle$
   c. Find an equation for the tangent plane at $(x, y, z) = \left(\dfrac{\sqrt{6}}{4}, \dfrac{\sqrt{6}}{4}, 1\right)$

4. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point (1,1,2).

5. Find the points on the surface $f(x, y) = \dfrac{y^2 - 9}{x}$ that are closest to the origin.

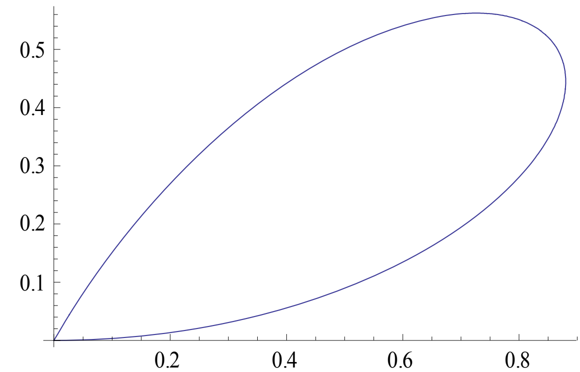
   Hint: In terms of the Lagrange multiplier method, this is equivalent to minimizing $x^2 + y^2 + z^2$ subject to the constraint $z = \dfrac{y^2 - 9}{x}$

6. Find the volume of the region below the surface $z = xe^y$ and above the portion of the disk $x^2 + y^2 = 25$ in the first quadrant.

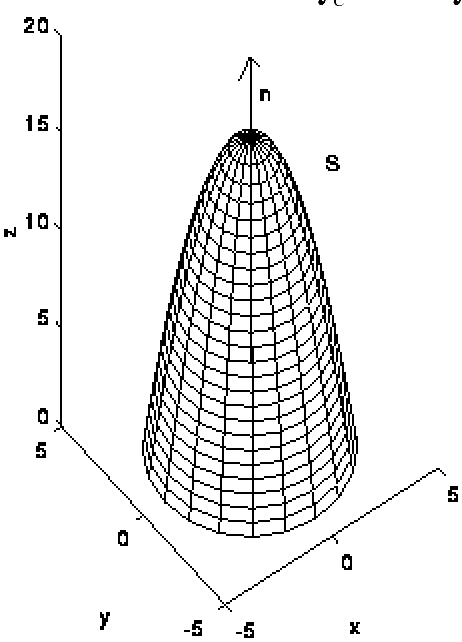
7. Find the flux of $\vec{F} = \langle x, y, -2z \rangle$ out of the surface S of the cube

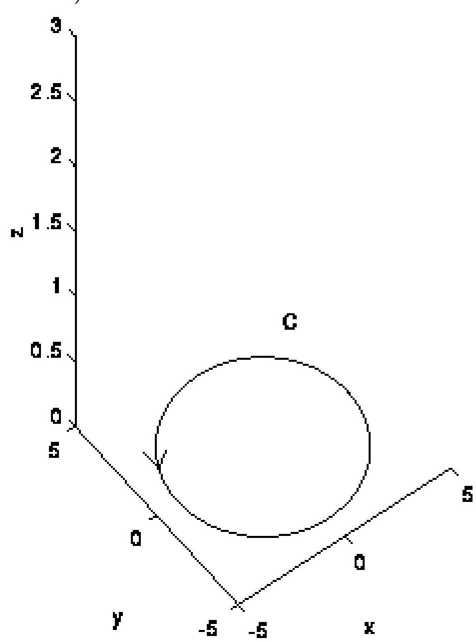
$C = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

8. Consider the region R in the first quadrant bounded by $r = \sin(3\theta)$
   a. Write $\iint_R \delta(r, \theta) dA$ as an iterated integral.
   b. Find the area of the region.
   c. Find the mass of the region with density function $\delta(r, \theta) = r$



9. Consider the surface described by the paraboloid $z = 16 - x^2 - y^2$ for $z \ge 0$, as shown below. Verify Stokes' Theorem for this surface and the vector field $\vec{F} = \langle 3y, 4z, -6x \rangle$. That is, evaluate both sides of the equation $\oint_C \vec{F} \cdot \overrightarrow{dr} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \overrightarrow{dS}$ and show they are equal.

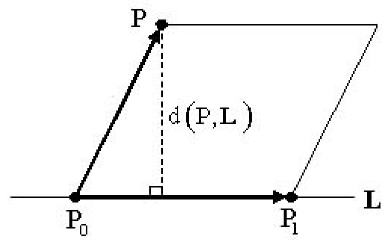




10. Use the Divergence Theorem to compute the surface integral $\iint_S \vec{F} \cdot \overrightarrow{dS}$ where the surface S is the sphere $x^2 + y^2 + z^2 = 1$ and the vector field is $\vec{F} = \langle x^3, y^3, z^3 \rangle$.

Math 2A – Vector Calculus – Final Exam Solutions – fall '09

1. Derive the formula for the distance d(P,L) between a point $P(x,y,0)$ and a line L through $P_0(x_0,y_0,0)$ and $P_1(x_1,y_1,0)$ in 3D by the following steps:



   a. Compute $\left|\overrightarrow{P_0P_1}\times\overrightarrow{P_0P}\right|$ in terms of x, y, x_0, y_0, and x_1, y_1.
   SOLN:

$$\left|\overrightarrow{P_0P_1}\times\overrightarrow{P_0P}\right| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1-x_0 & y_1-y_0 & 0 \\ x-x_0 & y-y_0 & 0 \end{vmatrix} = \left|\left\langle 0,0,(x_1-x_0)(y-y_0)-(y_1-y_0)(x-x_0)\right\rangle\right|$$

   b. Use the fact that the area of the parallelogram with edges $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P}$ is base*height to find a formula for the distance from P to L.
   SOLN: $(x_1-x_0)(y-y_0)-(y_1-y_0)(x-x_0)$ =

$$d(P,L)\left|P_0P_1\right| = d(P,L)\sqrt{(x_1-x_0)^2+(y_1-y_0)^2} = \left|(x_1-x_0)(y-y_0)+(y_1-y_0)(x-x_0)\right|$$

   So $d(P,L)=\dfrac{\left|(x_1-x_0)(y-y_0)+(y_1-y_0)(x-x_0)\right|}{\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}}$ If you let $a = x_1 - x_0$, $b = y_1 - y_0$,

   and $c = y_0(x_0 - x_1) + x_0(y_0 - y_1)$ then the formula becomes $d=\dfrac{|ay+bx+c|}{\sqrt{a^2+b^2}}$

2. Consider the curve $\vec{r}(t)=\langle 5\sin t, 12t, 5\cos t\rangle$

   a. Find the unit tangent vector as a function of t.

   SOLN: $\vec{r}\,'(t)=\langle 5\cos t,12,-5\sin t\rangle \Rightarrow \hat{T}(t)=\dfrac{\langle 5\cos t,12,-5\sin t\rangle}{13}$

   b. Find the curvature at $t = 0$.

   SOLN: $\kappa=\left|\dfrac{d\hat{T}}{ds}\right|=\left|\dfrac{d\hat{T}/dt}{ds/dt}\right|=\dfrac{\left|\langle -5\sin t,0,-5\cos t\rangle\right|}{\left|\vec{r}\,'(t)\right|^2}=\dfrac{5}{169}$ So the curvature is constant.

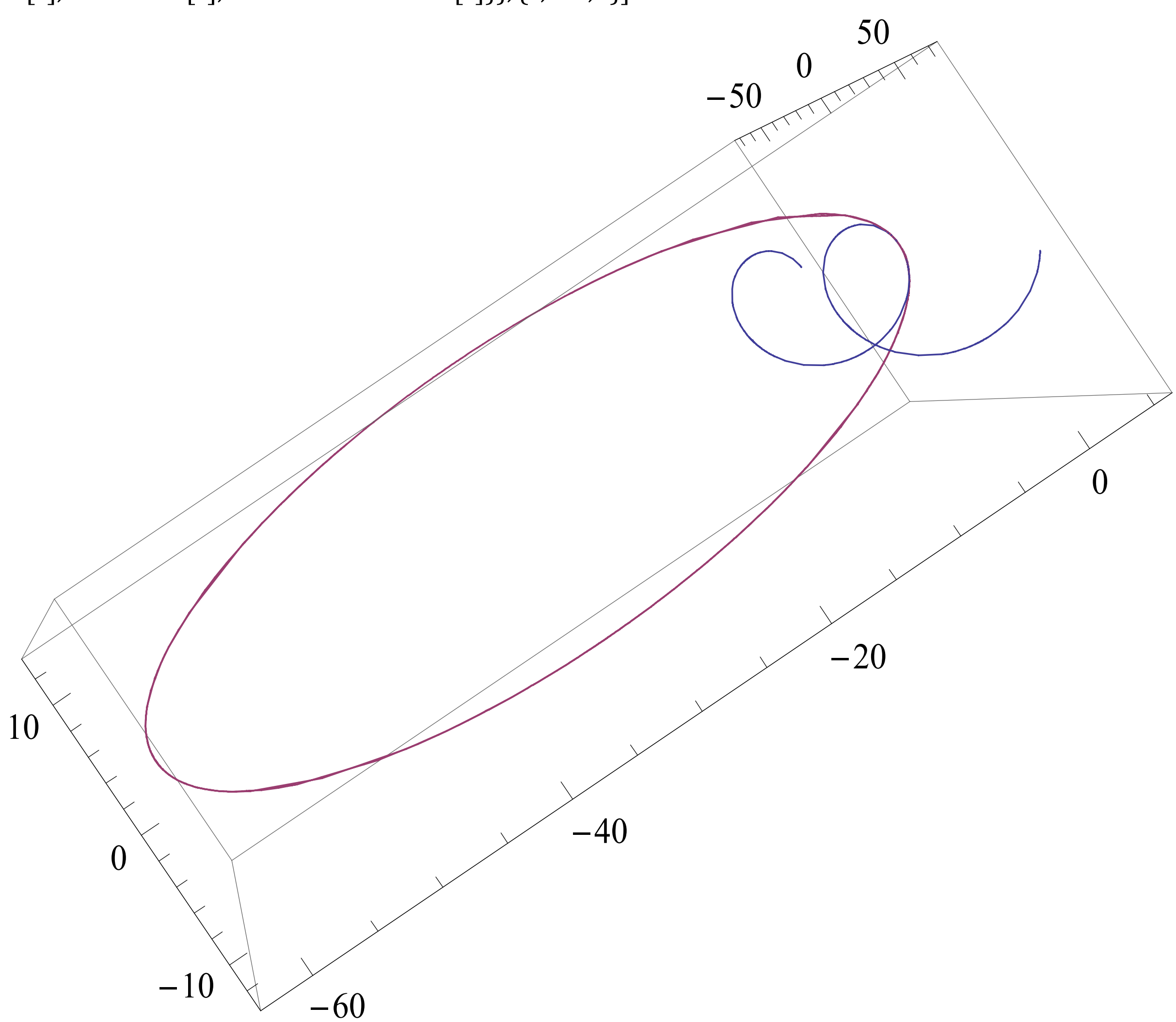
   c. Find a vector parallel to the normal to the curve at $t = 0$.

   SOLN: $\dfrac{d\hat{T}}{dt}=\langle -5\sin t,0,-5\cos t\rangle$ so $\langle -\sin t,0,-\cos t\rangle$ is parallel to the normal curve – oh, and it has length = 1, so it is the normal vector and at $t = 0$, $\hat{N}=\langle 0,0,-1\rangle$

   d. Parameterize the osculating circle at $t = 0$.

   SOLN: When $t = 0$, $\vec{r}(0)=\langle 0,0,5\rangle$, $\hat{N}=\langle 0,0,-1\rangle$ and the radius of the circle is 33.8, so the center of the circle is (0,0, –28.8) . To get the equation of the circle we need to stay in the osculating plane, which is spanned by $\hat{T}(0) = \frac{\langle 5,12,0 \rangle}{13}$ and $\hat{N} = \langle 0,0,-1 \rangle$ which shows that a normal to the osculating plane is the vector $\langle -12,5,0 \rangle$ so the osculating plane is the yz-plane where $x = 0$. Thus an equation for the osculating circle is the intersection of the plane $12x - 5y = 0$ with the sphere $x^2 + y^2 + (z + 28.8)^2 = (33.8)^2$. Solving the first equation for y and substituting into the second, we have $x^2 + \frac{144}{25}x^2 + \left(z + \frac{144}{5}\right)^2 = \frac{169^2}{25} \Leftrightarrow \left(\frac{x}{13}\right)^2 + \left(\frac{z + \frac{144}{25}}{\frac{169}{5}}\right)^2 = 1$, which can be parameterized by $\vec{u}(t) = \left\langle 13\cos t, \frac{156}{5}\cos t, \frac{169}{5}\sin t - \frac{144}{25} \right\rangle$.

To check, use the Mathematica code, ParametricPlot3D[{{5 * Sin[t],12 * t, 5 * Cos[t]}, {13 * Cos[t],31.2 * Cos[t], −28.8 + 33.8 * Sin[t]}}, {t, −6,6}]



3. Consider the ellipsoid $4x^2 + 4y^2 + z^2 = 4$.

   a. Show that this surface is described by the spherical function $\rho = \dfrac{2}{\sqrt{1+3\sin^2\phi}}$

   SOLN: $4x^2 + 4y^2 + z^2 = (x^2 + 3x^2) + (y^2 + 3y^2) + z^2 = x^2 + y^2 + z^2 + 3x^2 + 3y^2 = \rho^2 + 3r^2 = 4$ so that, substituting, $\rho = r\sin\phi$, we have $\rho^2(1 + 3(\sin\phi)^2) = 4$ and the result follows.

   b. Find a parameterization of the ellipsoid surface in terms of spherical coordinates θ and ϕ.
   Hint: recall that $\langle x, y, z\rangle = \langle \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi\rangle$

   SOLN: $\langle x, y, z\rangle = \left\langle \dfrac{2\cos\theta\sin\phi}{\sqrt{1+3\sin^2\phi}}, \dfrac{2\sin\theta\sin\phi}{\sqrt{1+3\sin^2\phi}}, \dfrac{2\cos\phi}{\sqrt{1+3\sin^2\phi}} \right\rangle$

   c. Find an equation for the tangent plane at $(x, y, z) = \left(\dfrac{\sqrt{6}}{4}, \dfrac{\sqrt{6}}{4}, 1\right)$

   SOLN: The simplest approach here would be to use gradient vector
   $\overrightarrow{\nabla f} = \langle 8x, 8y, 2z\rangle\Big|_{\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4}, 1\right)} = 2\langle \sqrt{6}, \sqrt{6}, 1\rangle$ as a normal so the equation is $\sqrt{6}x + \sqrt{6}y + z = 4$

4. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point (1,1,2).
   SOLN: It is sufficient to observe that the gradients are parallel. The gradient of the ellipsoid is $\overrightarrow{\nabla f} = \langle 6x, 4y, 2z\rangle\big|_{(1,1,2)} = \langle 6, 4, 4\rangle$ while the gradient of the sphere is
   $\overrightarrow{\nabla g} = \langle 2x-8, 2y-6, 2z-8\rangle\big|_{(1,1,2)} = -\langle 6, 4, 4\rangle$, so indeed they are parallel.

5. Find the points on the surface $f(x, y) = \dfrac{y^2 - 9}{x}$ that are closest to the origin.

   SOLN: The simpler way to do this, as I see it, is to substitute $z = \dfrac{y^2 - 9}{x}$ into the objective function and then find the critical points and check them with the second derivative test. Substituting, we get

   $f(x, y) = x^2 + y^2 + \dfrac{(y^2 - 9)^2}{x^2}$ so setting the partial derivatives to zero we get

   $f_x = 2x - \dfrac{2(y^2-9)^2}{x^3} = 0 \Leftarrow x^4 - (y^2-9)^2 = 0 \Leftrightarrow y^2 - 9 = \pm x^2$

   $f_y = 2y + \dfrac{4y(y^2-9)}{x^2} = 2y\left(1 + \dfrac{2(y^2-9)}{x^2}\right) = 0$. Thus either $y = 0$ or $1 + \dfrac{2(y^2-9)}{x^2} = 0$.

   Substituting $y^2 - 9 = \pm x^2$ we have $1 \pm \dfrac{2x^2}{x^2} = 0$, which isn't going to happen. So $y = 0$ and thus

   $(x,y) = (\pm 3, 0)$ are the critical points where $f(\pm 3, 0) = \dfrac{-9}{\pm 3} = \mp 3$ and the critical points are (3,0, –3) and (–3,0, 3). Now, it turns out the discriminant here is

$$f_{xx}f_{yy}-\left(f_{xy}\right)^2=\left(2+\frac{6\left(y^2-9\right)^2}{x^4}\right)\left(2+\frac{12y^2-36}{x^2}\right)-\left(-\frac{8y\left(y^2-9\right)}{x^3}\right)^2\Bigg|_{(x,y)=(\pm 3,0)}=(8)(-2)-0<0$$

so these are saddle points. Thus there is no minimum value. However, there is a lower bound on the set of distances. If you approach $x = 0$ along the paths $y^2 \pm x^2 = 9$ the $\lim_{x\to 0} z = \lim_{x\to 0}\frac{9\pm x^2-9}{x}=0$ you get to the point (0,3,0) which is a limiting point on the surface, but not actually on the surface, since $x = 0$. Thus 3 is a lower bound on the distance.

Alternatively, one could use Lagrange multipliers. You formulate the problem by saying you want to minimize $f(x,y,z)=x^2+y^2+z^2$ subject to the constraint $g(x,y,z)=z-\frac{y^2-9}{x}=0$. At the optimal point the tangent planes are parallel, so $\overrightarrow{\nabla f}=\lambda\overrightarrow{\nabla g}\Leftrightarrow\langle 2x,2y,2z\rangle=\lambda\left\langle\frac{y^2-9}{x^2},-\frac{2y}{x},1\right\rangle$

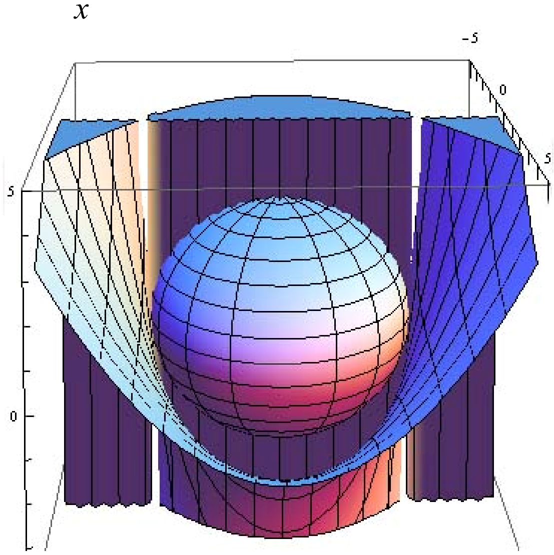
which leads to the four equations in 4 unknowns shown below. The second equation says that either $y = 0$ and/or $x = -\lambda$. If $x = -\lambda$ then the first equation has that $-2\lambda^2 = y^2 - 9$ so that, from the last equation, $z = 2\lambda$. Substituting into the third equation we'd have then that $4 = 1$, a contradiction. Thus $y = 0$. As the figures below illustrate the surface together with the sphere of radius 3 (first) where you can see a point of tangency is approached at (0,3,0) and the second where you can see the point of tangency at the saddle point (–3,0,3).

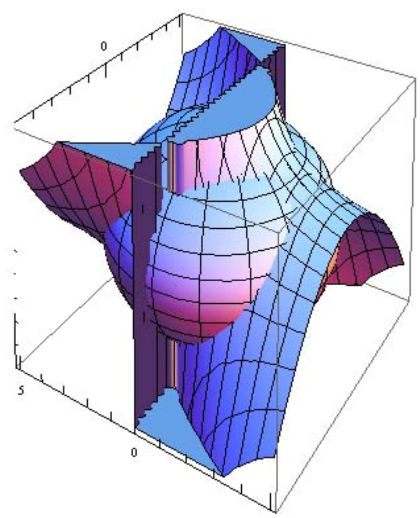
$$2x=\frac{\lambda\left(y^2-9\right)}{x^2}$$

$$y=-\frac{\lambda y}{x}$$

$$2z=\lambda$$

$$z=\frac{y^2-9}{x}$$





p7 = Plot3D[(y^2 − 9)/x , {x, −5,5}, {y, −5,5}, PlotRange → {−5,5}]

Finally, it should be noted that the surface $xz = y^2 - 9$ is a tilted hyperboloid of one sheet with removable discontinuities where $x = 0$. Maybe in Linear Algebra you'll learn to rotate it to perpendicular.

6. Find the volume of the region below the surface $z = xe^y$ and above the portion of the disk $x^2 + y^2 = 25$ in the first quadrant.

SOLN: Using cylindrical coordinates, $z = r\cos\theta\, e^{r\sin\theta}$ and the region is bounded by $0 \le r \le 5$ and $0 \le \theta \le \pi/2$ so the integral is $\int_0^5 \int_0^{\pi/2} r\cos\theta e^{r\sin\theta} r\,d\theta\,dr = \int_0^5 \int_0^1 r^2 e^{ru}\,du\,dr =$

$$\int_0^5 re^r - r\,dr = re^r\Big|_0^5 - \int_0^5 e^r\,dr - \frac{25}{2} = 5e^5 - e^5 + 1 - \frac{25}{2} = 4e^5 - \frac{23}{2}$$

7. Find the flux of $\vec{F} = \langle x, y, -2z \rangle$ out of the surface S of the cube

$C = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

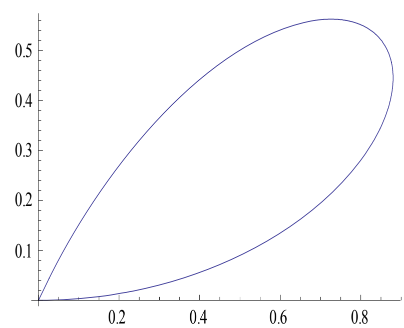
SOLN: The simplest way to work this is to use the divergence theorem and note that the divergence of $\vec{F}$ is $\vec{\nabla} \cdot \vec{F} = 1 + 1 - 2 = 0$ so the flux must be zero. To evaluate the surface integrals directly means $\oiint_S \vec{F} \cdot \overrightarrow{dS} = \iint_{x=0} \vec{F} \cdot \overrightarrow{dS} + \iint_{x=1} \vec{F} \cdot \overrightarrow{dS} + \iint_{y=0} \vec{F} \cdot \overrightarrow{dS} + \iint_{y=1} \vec{F} \cdot \overrightarrow{dS} + \iint_{z=0} \vec{F} \cdot \overrightarrow{dS} + \iint_{z=1} \vec{F} \cdot \overrightarrow{dS}$

$$= \iint_{x=0} \langle 0, y, -2z \rangle \cdot \langle -1, 0, 0 \rangle\, dA + \int_0^1 \int_0^1 \langle 1, y, z \rangle \cdot \langle 1, 0, 0 \rangle\, dydz$$

$$+ \iint_{y=0} \langle x, 0, -2z \rangle \cdot \langle 0, -1, 0 \rangle\, dA + \int_0^1 \int_0^1 \langle x, 1, z \rangle \cdot \langle 0, 1, 0 \rangle\, dxdz$$

$$+ \iint_{z=0} \langle x, y, 0 \rangle \cdot \langle 0, 0, -1 \rangle\, dA + \int_0^1 \int_0^1 \langle x, y, -2 \rangle \cdot \langle 0, 0, 1 \rangle\, dxdy = 0 + 1 + 0 + 1 + 0 - 2 = 0.$$

8. Consider the region R in the first quadrant bounded by $r = \sin(3\theta)$



a. Write $\iint_R \delta(r, \theta)\, dA$ as an iterated integral.

SOLN: $\int_0^{\pi/3} \int_0^{\sin 3\theta} \delta(r, \theta)\, r\,dr\,d\theta$

b. Find the area of the region.

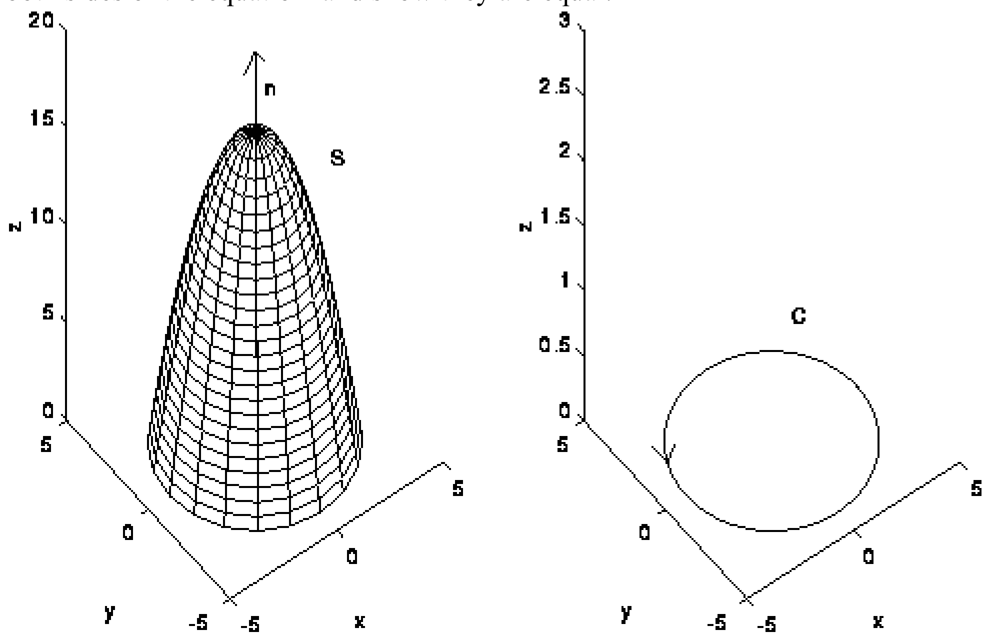
$$\int_0^{\pi/3} \int_0^{\sin 3\theta} r\,dr\,d\theta = \int_0^{\pi/3} \frac{\sin^2 3\theta}{2}\, d\theta = \int_0^{\pi/3} \frac{1 - \cos 6\theta}{4}\, d\theta = \frac{\pi}{12}$$

c. Find the mass of the region with density function $\delta(r, \theta) = r$

SOLN: $\int_0^{\pi/3} \int_0^{\sin 3\theta} r^2\,dr\,d\theta = \frac{1}{3} \int_0^{\pi/3} \sin^3 3\theta\, d\theta = \frac{1}{3} \int_0^{\pi/3} \sin 3\theta - \sin 3\theta \cos^2 3\theta\, d\theta$

$$= -\frac{1}{9} \cos 3\theta \Big|_0^{\pi/3} + \frac{1}{9} \int_1^{-1} u^3\, du = \frac{1}{9} + \frac{1}{9} + \frac{u^4}{36}\Big|_1^{-1} = \frac{2}{9}$$

9. Consider the surface described by the paraboloid $z = 16 - x^2 - y^2$ for $z \ge 0$, as shown below. Verify Stokes' Theorem for this surface and the vector field $\vec{F} = \langle 3y, 4z, -6x \rangle$. That is, evaluate both sides of the equation and show they are equal.



SOLN: $\vec{r}(t) = \langle 4\cos t, 4\sin t \rangle \Rightarrow \vec{r}\,'(t) = \langle -4\sin t, 4\cos t \rangle$ so that

$$\oint_C \vec{F} \cdot \overrightarrow{dr} = \int_0^{2\pi} \langle 12\sin t, 0, -24\cos t \rangle \langle -4\sin t, 4\cos t, 0 \rangle dt = -48\int_0^{2\pi} \sin^2 t dt = -48\pi$$

Now the surface is $\vec{r}(x,y) = \langle x, y, 16 - x^2 - y^2 \rangle$ so a measure of the scaled infinitesimal surface normal is $\overrightarrow{dS} = (\vec{r}_x \times \vec{r}_y) dxdy = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} dxdy = \langle 2x, 2y, 1 \rangle dxdy$ so that

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \overrightarrow{dS} = \iint_S \langle -4, 6, -3 \rangle \cdot \overrightarrow{dS} = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \langle -4, 6, -3 \rangle \cdot \langle 2x, 2y, 1 \rangle dxdy$$

$$= \int_0^4 \int_0^{2\pi} -8r^2\cos\theta + 12r^2\sin\theta - 3r d\theta dr = -6\pi\int_0^4 r dr = -48\pi$$

10. Use the Divergence Theorem to compute the surface integral $\iint_S \vec{F} \cdot \overrightarrow{dS}$ where the surface S is the sphere $x^2 + y^2 + z^2 = 1$ and the vector field is $\vec{F} = \langle x^3, y^3, z^3 \rangle$.

SOLN:L

$$\iint_S \vec{F} \cdot \overrightarrow{dS} = \iiint_E \vec{\nabla} \cdot \vec{F} dV = \iiint_E 3x^2 + 3y^2 + 3z^2 dV = 3\int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^4 \sin\phi d\phi d\theta d\rho = -\frac{6\pi}{5}\cos\phi\Big|_0^{\pi} = \frac{12\pi}{5}$$